PERGAMON

International Journal of Heat and Mass Transfer 46 (2003) 931–934

International Journal of HEAT and MASS TRANSFER

www.elsevier.com/locate/ijhmt

Technical Note

Explicit analytical solutions of 2-D laminar natural convection

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Received 2 February 2002; received in revised form 10 July 2002

Abstract

There are many natural convection processes in various fields, and it is still a hot topic to investigate the fluid dynamics and heat transfer of natural convection. The analytical solutions are meaningful in both theoretical investigation and practical applications. Specially, they are very useful to computational fluid dynamics and heat transfer as the benchmark solutions to check the numerical solutions and to develop numerical differencing schemes, grid generation methods and so forth. Two explicit analytical solutions of 2-D steady laminar natural convection along a vertical porous plate and between two vertical plates were derived for better understanding the flow and heat transfer as well as promoting the computational fluid dynamics and computational heat transfer.

Keywords: Analytical solution; Natural convection; Laminar flow; Boundary layer

1. Introduction

Natural convection widely exists in a variety of practical applications. In theoretical investigations of natural convection, analytical solutions are of significance. Many analytical solutions played a key role in the early development of fluid mechanics as well as for the heat conduction. However, the governing equations of natural convection flow are nonlinear and coupled, hence, it is highly difficult to obtain analytical solutions. By authors' knowledge, no new explicit analytical solution of natural convection flow has been found in the open literature for many years.

Besides their theoretical meaning, analytical solutions can also be applied to check the accuracy, convergence and effectiveness of various numerical computation methods and to improve their differencing schemes, grid generation ways and so on. The analytical solutions are therefore very useful even for the newly rapidly developing computational fluid dynamics and heat transfer. For example, several analytical solutions which can simulate the 3-D potential flow in turbomachine cascades were obtained by Cai et al. [1], and were successfully used by some investigators in their numerical calculation to check their computational techniques and computer codes [1–4]. In addition, authors recently presented some explicit analytical solutions of unsteady compressible flow and heat transfer [5–14]. In this paper, two algebraically explicit analytical solutions of 2-D steady laminar natural convection are derived to develop the theoretical understanding, and also, to serve as the benchmark solutions for numerical calculations. The derivation procedure in this paper is mainly based on the method of separation variables with addition employed by the authors in previous researches [7–14]. In this method, to separate an unknown function $f(x,y)$ with assumption $f = X(x) + Y(y)$ instead of $f = X(x) \cdot Y(y)$ in common method. However, for a given analytical solution, its correctness can be proven easily by substituting it into the governing equations.

2. Governing equations

The governing equations for 2-D steady laminar natural convection with constant kinematic viscosity and thermal diffusivity can be expressed as follows

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PII: S0017-9310(02)00359-9

Nomenclature

a	thermal diffusivity
C	constant
u	velocity component in x direction
v	velocity component in y direction
x	abscissa
y	coordinate
β	coefficient of expansion
θ	temperature
υ	kinematic viscosity
Ψ	stream function
Subscripts	
$0, 1, 2, 3\ldots$	different constant
∞	free stream

(neglecting dissipation heat, radiation and internal heat source. The x-direction is opposite to the gravitation) [15],

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1a}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\beta g(\theta-\theta_\infty)+\upsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{1b}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\upsilon\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \tag{1c}$$

$$u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}=a\left(\frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}\right) \tag{1d}$$

With boundary layer assumption, the governing equations can be simplified as,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2a}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\beta g(\theta-\theta_\infty)+\upsilon\frac{\partial^2 u}{\partial y^2} \tag{2b}$$

$$u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}=a\frac{\partial^2 \theta}{\partial y^2} \tag{2c}$$

If the solutions of boundary equations (2) satisfy following conditions: $v=\text{const.}$, $u=u(y)$ and $\theta=\theta(y)$, they are actually the solutions of original equations (1) without boundary layer assumption.

3. Solution of natural convection along infinite vertical porous plate

If the case of boundary layer is firstly considered and the stream function ψ with $u=\partial\psi/\partial y$ and $v=-(\partial\psi/\partial x)$ is introduced, the equation set (2) evolves to (the continuity equation is satisfied already),

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=\beta g(\theta-\theta_\infty)+\upsilon\frac{\partial^3\psi}{\partial y^3} \tag{3a}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}=a\frac{\partial^2\theta}{\partial y^2} \tag{3b}$$

Assuming

$$\psi=X_1(x)+Y_1(y),\quad u=Y_1'(y)\quad\text{and}\quad v=-X_1'(x) \tag{4}$$

$$\theta=X_2(x)+Y_2(y) \tag{5}$$

yields the following equations derived from equation set (3),

$$-X_1'Y_1''=\beta g(X_2+Y_2-\theta_\infty)+\upsilon Y_1''' \tag{6a}$$

$$Y_1'X_2'-X_1'Y_2'=aY_2'' \tag{6b}$$

If the following condition further holds true

$$X_1=C_1x,\quad X_1'=C_1,\quad v=-C_1 \tag{7}$$

the equation set (6a) and (6b) are simplified as,

$$-C_1Y_1''=\beta g(X_2+Y_2-\theta_\infty)+\upsilon Y_1''' \tag{8a}$$

and

$$Y_1'X_2'-C_1Y_2'=aY_2'' \tag{8b}$$

The variables of Eqs. (8a) and (8b) can be separated,

$$X_2-\theta_\infty=C_2=-(C_1Y_1''+\upsilon Y_1'''+\beta gY_2)/(\beta g) \tag{9a}$$

$$X_2'=(aY_2''+C_1Y_2')/Y_1' \tag{9b}$$

From the left-hand side of Eq. (9a), we have

$$X_2=C_2+\theta_\infty\quad\text{and}\quad X_2'=0 \tag{10}$$

Substituting Eq. (10) into Eq. (9b), the expression of Y_2 is derived as,

$$Y_2=-aC_3\mathrm{e}^{-(C_1/a)y}/C_1 \tag{11}$$

With substitution of Eq. (11) into the right-hand side of Eq. (9a), the expression of $u=Y_1'$ is deduced as,

$$u=Y_1'=-C_3\beta ga^2\mathrm{e}^{-(C_1/a)y}/[C_1^3(1-\upsilon/a)]-C_4\upsilon\mathrm{e}^{-(C_1/\upsilon)y}/C_1-C_2\beta gy/C_1+C_5 \tag{12}$$

and the expression of θ is obtained by summing up Eqs. (10) and (11):

$$\theta=X_2+Y_2=\theta_\infty+C_2-aC_3\mathrm{e}^{-(C_1/a)y}/C_1 \tag{13}$$

Therefore, an algebraically explicit analytical solution of governing equation set (2) is obtained, the velocity field Eqs. (7) and (12) as well as the temperature field Eq. (13).

When $C_1 > 0$, $C_2 = 0$ and $C_5 = C_3\beta g a^2/[C_1^3(1-\upsilon/a)] + C_4\upsilon/C_1$, this solution satisfies the boundary layer condition along an infinite vertical cold porous plate,

$$y = 0:\ u = 0, \quad v = -C_1 \quad \text{and} \quad \theta = \theta_\infty - C_3 a/C_1$$

$$y = \infty:\ u = C_5, \quad \frac{\partial u}{\partial y} = 0 \quad \text{and} \quad \theta = \theta_\infty, \quad \frac{\partial \theta}{\partial y} = 0$$

It represents the natural convection in a semi-infinite space with boundary suction along an infinitely vertical long cold porous plate. The horizontal velocity and the suction velocity are constant, $v = -C_1$ in whole space. Outside the boundary layer, both vertical velocity and temperature are almost constant, or $u \approx C_5$ and $\theta \approx \theta_\infty$. Owing to the suction effect, the boundary layer thinkness is constant also and all parameters are functions of coordinate y only due to the combined effects of suction and cold natural convection.

The physical description of this solution and the variations of u and θ along y are given in Fig. 1 (the vectors v are out of proportion to vectors u).

It should be emphasized that the parameters of this solution are functions of y only and v is constant. So, the solution is not only the solution of boundary layer but also the solution of general laminar natural convection equations.

When $C_2 \neq 0$, and C_5 still equals to $C_3\beta g a^2/\ [C_1^3(1-\upsilon/a)] + C_4\upsilon/C_1$, the solution represents the natural convection between two parallel infinite vertical porous plates. The right plate is cold and steady and similar to the above mentioned case while the left plate is warmer with $\theta \approx \theta_\infty + C_2$ when the distance between two plates is wide enough, and it moves upward or downward with a speed

$$u_w \approx C_5 - C_2\beta g d/C_1 - C_3\beta g a^2 e^{-(C_1/a)d}/[C_1^3(1-\upsilon/a)] - C_4\upsilon e^{-(C_1/\upsilon)d}/C_1$$

where d is the distance between two plates.

The physical description of this case is shown in Fig. 2. When

$$C_2 = C_1\{C_5 - [-C_3\beta g a^2 e^{-(C_1/a)d}/[C_1^3(1-\upsilon/a)] - C_4\upsilon e^{-(C_1/\upsilon)d}/C_1]\}/(\beta g d)$$

both plates are steady. Similar to previous case, there is a horizontal velocity $v = -C_1$ through both porous plates also.

If $C_1 < 0$, the solution represents the laminar natural convection between two parallel infinite vertical porous plates having different temperatures moving up or down (determined by the constants) with different speeds. One simple example is shown in Fig. 3.

When $C_5 = C_3\beta g a^2/[C_1^3(1-\upsilon/a)] + C_4\upsilon/C_1$, the right plate is steady.

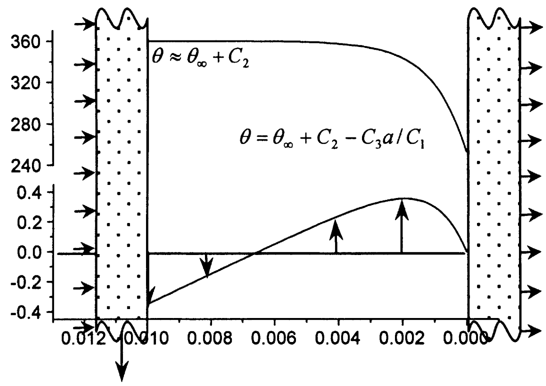

Fig. 2. The physical description of Eqs. (7), (12) and (13) with $C_1 > 0$ and $C_2 \neq 0$.

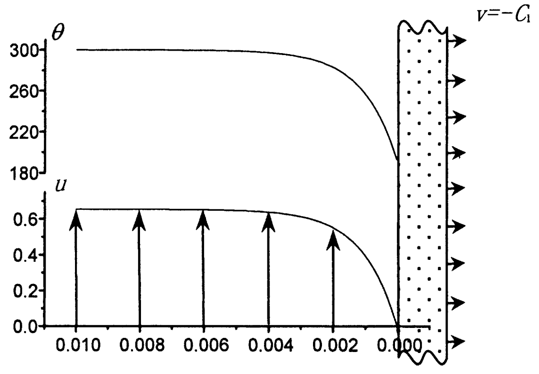

Fig. 1. The physical description of Eqs. (7), (12) and (13) with $C_1 > 0$ and $C_2 = 0$.

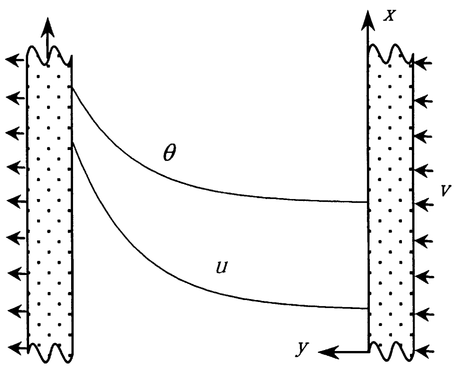

Fig. 3. The physical description of Eqs. (7), (12) and (13) with $C_1 < 0$.

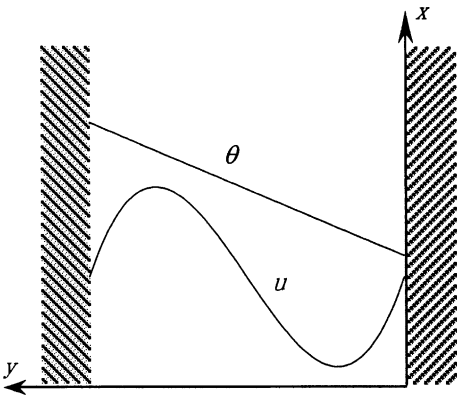

Fig. 4. The physical description of Eqs. (14).

4. Solution of natural convection between two infinite solid plates

In the derivation procedure of previous paragraph, if $C_1 = 0$, another algebraically explicit analytical solution can be easily deduced with Eqs. (9a) and (9b) as,

$$\left.\begin{aligned} u &= \beta g[-C_0 y^3/6 + (\theta_\infty - C_2)y^2/2 + C_3 y + C_4]/\upsilon \\ v &= 0 \\ \theta &= C_0 y + C_2 \end{aligned}\right\} \tag{14}$$

This solution describes the natural convection between two infinite parallel vertical solid plates. Generally, both plates are moving. However, the right plate ($y = 0$) will be steady when $C_4 = 0$; and furthermore, the left plate ($y = 1$) will be steady also when $C_3 = C_0/6 + (C_2 - \theta_\infty)/2$. The temperature linearly increases along horizontal direction when $C_0 > 0$.

Similar to previous situation, the solution is a solution of general laminar natural convection satisfying equation set (1).

A graphical illustration of this solution is shown in Fig. 4. The natural convection introduced by left hot plate and right cold plate is evident.

5. Summary

Two new algebraically explicit analytical solutions of 2-D steady laminar natural convection are theoretically obtained. By authors' knowledge, no such analytical solutions are available in the open literature so far. These solutions are valuable to the physical significance for understanding of natural convection, especially to the computational heat transfer as the benchmark solutions to check the numerical solutions and to develop the numerical computation approaches such as the differencing schemes, grid generation methods and so forth.

Acknowledgements

The study is supported by the National Science Foundation of China (50246003 and 59925615) and NKBRSF Project of China (G2000026305 and G19990223).

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